

Towards the exact dilatation operator of $\mathcal{N} = 4$ super Yang-Mills theory

A.V. Ryzhov^{1,*} and A.A. Tseytlin^{2,†}

¹ *Department of Physics, Brandeis University
Waltham, MA 02454, USA*

² *Department of Physics, The Ohio State University
Columbus, OH 43210-1106, USA*

Abstract

We investigate the structure of the dilatation operator D of planar $\mathcal{N} = 4$ SYM in the sector of single trace operators built out of two chiral combinations of the 6 scalars. Previous results at low orders in 't Hooft coupling λ suggest that D has a form of an $SU(2)$ spin chain Hamiltonian with long range multiple spin interactions. Instead of the usual perturbative expansion in powers of λ , we split D into parts $D^{(n)}$ according to the number n of independent pairwise interactions between spins at different sites. We determine the coefficients of spin-spin interaction terms in $D^{(1)}$ by imposing the condition of regularity of the BMN-type scaling limit. For long spin chains, these coefficients turn out to be expressible in terms of hypergeometric functions of λ , which have regular expansions at both small and large values of λ . This suggests that anomalous dimensions of “long” operators in the two-scalar sector should generically scale as $\sqrt{\lambda}$ at large λ , i.e. in the same way as energies of semiclassical states in dual $AdS_5 \times S^5$ string theory.

*E-mail: ryzhovav@brandeis.edu

†Also at Imperial College London and Lebedev Institute, Moscow

1 Introduction

The $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills theory [1] is the basic example of a 4-d conformal theory [2]. It is actually a family of CFT's parametrized by N and the 't Hooft coupling $\lambda = g_{\text{YM}}^2 N$. To solve a CFT amounts, at least, to being able to compute dimensions of local gauge invariant conformal operators as functions of the parameters. This problem should simplify in the planar limit of $N \rightarrow \infty$, λ fixed. In this limit the AdS/CFT duality conjecture [3] suggests that conformal dimensions should be smooth functions of λ , and have regular expansions at both large and small λ .

Important progress towards this nontrivial goal of understanding how anomalous dimensions depend on λ was recently made by concentrating on states with large quantum numbers (see, in particular, [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21]). For some non-BPS states there are new expansion limits (like large J , fixed $\tilde{\lambda} \equiv \frac{\lambda}{J^2}$ for S^5 -rotating pointlike [4] and multispin [12] string states) where one can directly compare perturbative SYM anomalous dimensions to semiclassical string results.

One would obviously like to go beyond the restriction to long and/or scalar-only operators and compute, e.g., the exact dimension of the Konishi scalar $\text{tr}(\Phi_i^* \Phi_i)$, or the coefficient $f(\lambda)$ of the $\ln S$ term in the anomalous dimension of the minimal twist operators such as $\text{tr}(\Phi_i^* D^S \Phi_i)$ [5, 8, 22]. The main obstacle is our lack of tools for obtaining exact all-order results on either gauge theory ($\sum c_n \lambda^n$) or string theory ($\sum \frac{b_n}{(\sqrt{\lambda})^n}$) side. One potentially fruitful idea of how to go beyond the first few orders in SYM perturbation theory is to try to determine the exact structure of the dilatation operator D by imposing additional conditions (like superconformal symmetry, BMN limit, integrability, etc., as in [23, 24, 25]) implied by the expected correspondence with $AdS_5 \times S^5$ string theory. Having found the resulting anomalous dimensions as functions of λ , one may then be able to see if they admit a regular expansion not only at small but also at large λ .

This is the approach we would like to explore below using as an input the condition of regularity of BMN-type scaling limit in the form suggested in [18] and further clarified in [21]. We shall concentrate on the planar $SU(2)$ sector of single trace SYM operators built out of chiral combinations X and Z of two the 6 SYM adjoint scalars, i.e. $\text{tr}(X \dots X Z \dots Z X \dots)$ with canonical dimension L . This sector is closed under renormalization [23]. The eigen-operators of D with J_1 Z 's and J_2 X 's (so $L = J_1 + J_2$) should be

dual to string states with two components of the $SO(6)$ spin [4, 12]. On general grounds, the SYM dilatation operator computed in the planar limit should be a series in λ

$$D = \sum_{r=0}^{\infty} \frac{\lambda^r}{(4\pi)^{2r}} D_{2r} . \quad (1.1)$$

Let us review what is known already about the structure of D_{2r} .

Restricting D to planar graphs suggests that D_{2r} should be given by local sums over sites $a = 1, \dots, L$ with Z and X interpreted as a spin “up” and spin “down” state of a periodic $(a + L \equiv a)$ spin chain [11, 23] for which D is the Hamiltonian,

$$D_{2r} = \sum_{a=1}^L \mathcal{D}_{2r}(a) , \quad \mathcal{D}_0 = \mathbf{1} . \quad (1.2)$$

The one-loop term \mathcal{D}_2 turns out to be equivalent to the Hamiltonian of the ferromagnetic $XXX_{\frac{1}{2}}$ Heisenberg spin chain [11],

$$\mathcal{D}_2 = 2Q_{a,a+1} , \quad (1.3)$$

$$Q_{a,b} \equiv \mathbf{1} - P_{a,b} , \quad \text{i.e.} \quad Q_{a,b} = \frac{1}{2}(\mathbf{1} - \vec{\sigma}_a \cdot \vec{\sigma}_b) , \quad (1.4)$$

where $P_{a,b}$ is the permutation operator and $\vec{\sigma}_a$ are the Pauli matrices acting on the spin state at site a .¹ The two-loop term \mathcal{D}_4 was found to be [23]

$$\mathcal{D}_4 = 2(-4Q_{a,a+1} + Q_{a,a+2}) , \quad (1.5)$$

while the expression for the 3-loop term \mathcal{D}_6 conjectured in [23] on the basis of integrability

$$\begin{aligned} \mathcal{D}_6 = & 4(15Q_{a,a+1} - 6Q_{a,a+2} + Q_{a,a+3}) \\ & + 4(Q_{a,a+2}Q_{a+1,a+3} - Q_{a,a+3}Q_{a+1,a+2}) , \end{aligned} \quad (1.6)$$

was shown in [25] to be uniquely fixed by the superconformal symmetry algebra, constraints coming from the structure of Feynman graphs and the correct BMN limit.² Finally, there is also a proposal [24] for the 4-loop term \mathcal{D}_8 based on assuming integrability

¹For $a = b$ one should set $Q_{a,a} = 0$; note that $P^2 = \mathbf{1}$ and $\frac{1}{2}Q$ is a projector.

²The same expression was found in a closely related context of SYM matrix model [26]. Also, the 3-loop anomalous dimension of (a descendant) of the Konishi operator found in [23] from the above form of D_6 received a remarkable indirect confirmation in a recent computation of anomalous dimension in twist 2 sector [22] which also contains a descendant of the Konishi operator (N. Beisert and M. Staudacher, private communication).

[23] and the BMN scaling. Written in terms of factorized permutations as in [19] it reads

$$\begin{aligned}
\mathcal{D}_8 = & 10(-56Q_{a,a+1} + 28Q_{a,a+2} - 8Q_{a,a+3} + Q_{a,a+4}) \\
& + \frac{2}{3} \left[(421Q_{a,a+1}Q_{a+2,a+3} + 986Q_{a,a+3}Q_{a+1,a+2} - 183Q_{a,a+2}Q_{a+1,a+3}) \right. \\
& + 8(Q_{a,a+3}Q_{a+2,a+4} + Q_{a,a+2}Q_{a+1,a+4} - Q_{a,a+4}Q_{a+2,a+3} - Q_{a,a+4}Q_{a+1,a+2} \\
& \left. - Q_{a,a+4}Q_{a+1,a+3} + Q_{a,a+3}Q_{a+1,a+4}) \right] . \tag{1.7}
\end{aligned}$$

Generalizing the above expressions for $r = 2, 3, 4$ it is then natural to expect that generic r -loop term in (1.2) will contain a term linear in $Q_{a,b}$, a term quadratic in $Q_{a,b}$, and so on:

$$\mathcal{D}_{2r} = \mathcal{D}_{2r}^{(1)} + \mathcal{D}_{2r}^{(2)} + \dots , \quad \mathcal{D}_{2r}^{(n)} \sim \sum Q^n , \tag{1.8}$$

$$\mathcal{D}_{2r}^{(1)} = 2 \sum_{c=1}^r a_{r,c} Q_{a,a+c} . \tag{1.9}$$

At order r there can be at most r spin-spin interactions in $\mathcal{D}_{2r}^{(1)}$ [6, 23]. Q^n in $\mathcal{D}_{2r}^{(n)}$ stands for products of independent projectors, i.e. with all indices corresponding to different sites as in (1.6) and (1.7). The above explicit expressions (1.3), (1.5), (1.6), (1.7) for D_2, \dots, D_8 imply that for $r \leq 4$ the coefficients $a_{r,c}$ are ($c = 1, 2, \dots, r$)

$$a_{1,1} = 1 ; \quad a_{2,c} = (-4, 1) ; \quad a_{3,c} = (30, -12, 2) ; \quad a_{4,c} = (-280, 140, -40, 5) . \tag{1.10}$$

Then

$$D = D_0 + D^{(1)} + D^{(2)} + \dots , \quad D^{(1)} = 2 \sum_{r=1}^{\infty} \frac{\lambda^r}{(4\pi)^{2r}} \sum_{a=1}^L \sum_{c=1}^r a_{r,c} Q_{a,a+c} . \tag{1.11}$$

Using the periodicity of the chain ($Q_{a,b+L} = Q_{a,b}$, etc.) $D^{(1)}$ can be rewritten as

$$D^{(1)} = \sum_{a=1}^L \sum_{c=1}^{L-1} h_c(L, \lambda) Q_{a,a+c} . \tag{1.12}$$

Our aim below will be to determine the general expression for the coefficients $a_{r,c}$ and thus the functions $h_c(L, \lambda)$, i.e. to find the spin-spin (linear in Q) part of the exact dilatation operator D .

To find the coefficients $a_{r,c}$ in (1.9), we will demand that the BMN-type scaling limit

$$L \rightarrow \infty , \quad \tilde{\lambda} \equiv \frac{\lambda}{L^2} = \text{fixed} \tag{1.13}$$

of the coherent-state expectation value of $\mathcal{D}^{(1)}$ [18, 21] is well defined. This turns out to be (nearly) equivalent to the consistency with the BMN expression [4, 6, 7] for the

anomalous dimensions of the 2-impurity operators. Imposing the condition of agreement with the BMN square root formula fixes one remaining free coefficient at each order in r .

Our approach is thus similar to the previous important investigations of the constraints on the dilatation operator imposed by the BMN limit [6, 23, 24, 25]. The new elements of the present discussion are that (i) we follow [19, 21] and classify the structures in D in terms of independent interactions between sites as in (1.8),(1.11), i.e. $D = D_0 + \sum Q + \sum QQ + \dots$, and (ii) we resum the loop expansion to find the coefficients in $D^{(1)} = \sum Q$ as explicit functions of λ and study their strong-coupling limit.

The resulting $D^{(1)}$ (1.12) may be interpreted as a Hamiltonian of a periodic spin chain with long-range interactions. One could conjecture that, like in some known examples [27, 28], this spin chain may be integrable; and, furthermore, the higher-order terms $D^{(2)}, D^{(3)}, \dots$ in (1.11) may be effectively determined by $D^{(1)}$, e.g., expressed in terms of higher conserved charges of the chain. This would then determine the full D . Remarkably, this is indeed true up to order λ^3 [19]: the sum of one, two, and three-loop dilatation operators can be viewed as a part of the Inozemtsev integrable spin chain [28], with the QQ -terms in D_6 in (1.6) being proportional to a leading term in the λ -expansion of a higher conserved charge of the Inozemtsev chain. At order λ^4 the Hamiltonian of the Inozemtsev chain does not, however, agree with the BMN perturbative scaling [19]. Here, instead of starting with the Inozemtsev chain, we reverse the logic and determine which spin chain Hamiltonian is actually consistent with the BMN limit, leaving the issue of integrability open.³

For comparison, let us recall that the Inozemtsev Hamiltonian that interpolates between the Heisenberg and Haldane-Shastry spin chain Hamiltonians is given by [28]

$$H = \sum_{a=1}^L \sum_{c=1}^{L-1} p_c(L, q) Q_{a, a+c} , \quad (1.14)$$

where $p_c(L, q)$ is a double-periodic Weierstrass function with periods L and q

$$p_c = \frac{1}{c^2} + \sum_{(m,n) \neq (0,0)} \left[\frac{1}{(c - mL - inq)^2} - \frac{1}{(mL + inq)^2} \right] . \quad (1.15)$$

Note that $p_c = p_{L-c}$, so the sum in (1.14) may be restricted to $c \leq [L/2]$. The limiting cases are $\lim_{q \rightarrow \infty} p_c = (\frac{\pi}{L})^2 (\frac{1}{\sin^2 \frac{\pi c}{L}} - \frac{1}{3})$ (the Haldane-Shastry chain limit), and $\lim_{L \rightarrow \infty} p_c =$

³Footnote 5 in [19] points out, following [28], that the only integrable spin chain with spin-spin interactions is the Inozemtsev chain. This implies that higher-order terms $\mathcal{D}^{(n)}$ may not be directly determined by $\mathcal{D}^{(1)}$, and only the whole D may represent a Hamiltonian of an integrable spin chain.

$(\frac{\pi}{q})^2(\frac{1}{\sinh^2 \frac{\pi c}{q}} + \frac{1}{3})$ ($q \rightarrow 0$ corresponds to the infinite Heisenberg chain). As was suggested in [19], to relate H in (1.14) to the 3-loop SYM dilatation operator one is to relate q to λ by $\frac{\lambda}{(4\pi)^2} = \sum_{n=1}^{\infty} (4 \sinh^2 \frac{\pi n}{q})^{-1}$.

The rest of this paper is organized as follows. In section 2 we determine the coefficients in (1.9) by first imposing the regularity of the scaling limit (1.13) of the coherent-state expectation value of $D^{(1)}$ following [18, 21] (section 2.1) and then checking consistency with the 2-impurity BMN spectrum (section 2.2). In principle, we could fix all the coefficients just by imposing the second (BMN) condition but we believe the approach of section 2.1 is more straightforward and has its own conceptual merit, having close connection to string theory [18, 21].

In section 3 we discuss how to sum up the 't Hooft coupling expansion of the coefficients in $D^{(1)}$. We first show that in the large L limit the coefficients of $Q_{a,a+c}$ can be expressed in terms of hypergeometric functions which smoothly interpolate between perturbative power series at small λ and $\sqrt{\lambda}$ growth at large λ . We then comment on the finite L case, in particular on the resulting contribution of $D^{(1)}$ to the exact anomalous dimension of the Konishi operator.

Section 4 contains some conclusions and a discussion of open problems. Some useful relations and definitions are summarized in Appendix.

2 Determining the coefficients in $D^{(1)}$

Our first task will be to obtain the general expression for the coefficients $a_{r,c}$ in the linear in $Q_{a,a+c}$ part of the dilatation operator in (1.2),(1.9). To do this we shall follow [18, 21] and consider the spin coherent state path integral representation for the quantum mechanics of D (1.1) as a generalized spin chain Hamiltonian. The corresponding action for a collection of unit 3-vectors $\vec{n}_a(t)$ at each of L sites of the chain ($\langle n | \sigma_a^i | n \rangle = n_a^i$, $(n_a^i)^2 = 1$, $i = 1, 2, 3$) will then be given by

$$S = \int dt \left[\sum_{a=1}^L \mathcal{L}_{\text{WZ}}(n_a) - \langle D \rangle \right], \quad \langle D \rangle = \langle n | D | n \rangle, \quad (2.1)$$

where $\mathcal{L}_{\text{WZ}}(\vec{n}_a) = C_i(\vec{n}_a)\dot{n}_a^i$ ensures the proper $SU(2)$ commutation relations if one considers \vec{n}_a back as spin operators (see, e.g., [29]). Motivated by the explicit results (1.3)–(1.7) where the leading (at small λ) coefficient of $Q_{a,a+c}$ is always positive, it is natural to assume that the spin chain in question is ferromagnetic.⁴ Then in the long spin chain limit $L \rightarrow \infty$ one may expect [18, 21] that the low energy excitations of the spin chain will be captured by the semiclassical dynamics of (2.1). The correspondence with string theory then suggests [18, 21] that S should have a regular scaling limit (1.13), or, more explicitly, that the low energy effective action for the system governed by (2.1) should have a well defined continuum limit. To take the continuum limit one may introduce a field $\vec{n}(\sigma, t)$, $0 < \sigma \leq 2\pi$, with $\vec{n}_a(t) = \vec{n}(\frac{2\pi a}{L}, t)$, so that (2.1) becomes

$$S \rightarrow L \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[C_i(\vec{n})\dot{n}_a^i - \mathcal{H}(\partial\vec{n}, \partial^2\vec{n}, \dots; \tilde{\lambda}) \right]. \quad (2.2)$$

\mathcal{H} which originated from $\langle D \rangle$ should be a regular function of the effective coupling $\tilde{\lambda}$ and σ -derivatives of $\vec{n}(t, \sigma)$ in the limit $L \rightarrow \infty$, $\tilde{\lambda}$ fixed (with subleading $\frac{1}{L^n}$ terms omitted). Quantum corrections are then suppressed because of the large prefactor L in front of the action.

Writing D in terms of factorized permutation operators as in (1.5)–(1.7) one observes that since the $Q_{a,b}$ in (1.4) satisfy

$$\langle n | Q_{a,b} | n \rangle = \frac{1}{2}(1 - \vec{n}_a \cdot \vec{n}_b) = \frac{1}{4}(\vec{n}_a - \vec{n}_b)^2, \quad (2.3)$$

$\langle n | \mathcal{D}^{(1)} | n \rangle$ in (1.8) contains terms *quadratic* in \vec{n} (but all orders in derivatives); $\langle n | \mathcal{D}^{(2)} | n \rangle$, terms *quartic* in n , etc. The approximation that distinguishes $\mathcal{D}^{(1)}$ from all higher $\mathcal{D}^{(k)}$ in (1.8) is the one in which one keeps only small fluctuations of $\vec{n}(t, \sigma)$ near its (“all-spins-up”) ground-state value $\vec{n}_0 = (0, 0, 1)$. Then $\vec{n} = \vec{n}_0 + \vec{\delta n}$, where $|\vec{\delta n}| \ll 1$, so that higher powers of the fluctuating field $\vec{\delta n}$ are suppressed, regardless the number of spatial derivatives acting on them. Such configurations correspond to semiclassical spinning string states with $J_1 \gg J_2$, and are close to a single-spin BPS state. They should indeed represent semiclassical or coherent-state analogs of few-impurity BMN states, having the same BMN energy-spin relation which is indeed reproduced in the limit $J_1 \gg J_2$ [8, 12, 14, 15] by the classical two-spin string solutions.

Let us therefore consider this BMN-type approximation, concentrating on the part of $\langle n | D | n \rangle$ which is quadratic in n , i.e. on $\langle n | \mathcal{D}^{(1)} | n \rangle$, and demand that the continuum

⁴Then the state with all spins “up” (represented by the operator $\text{tr } Z^L$) is a true vacuum.

version of $\langle n|D^{(1)}|n\rangle$ have a regular scaling limit (1.13). As was shown in [18, 21], this condition is indeed satisfied at $r = 1, 2, 3$ loop orders, i.e. for (1.3), (1.5) and (1.6) (in general, this should be a consequence of the supersymmetry of the underlying SYM theory which restricts the structure of D). One finds that the coefficients in the order Q -terms in (1.5)–(1.7) are such that all lower than r -derivative terms in the continuum limit of $\langle n|\mathcal{D}_r^{(1)}|n\rangle$ cancel out. If they did not, the limit $L \rightarrow \infty$ would be singular, as there would be a disbalance between the powers of λ and the powers of L . Explicitly, one gets for $r = 1, 2, 3, 4$ using (1.10) (after integrating by parts)

$$\frac{\lambda^r}{(4\pi)^{2r}} \langle \mathcal{D}_{2r}^{(1)} \rangle = \frac{\lambda^r}{2(4\pi)^{2r}} \sum_{c=1}^r a_{r,c} (\vec{n}_a - \vec{n}_{a+c})^2 \rightarrow d_r \tilde{\lambda}^r \left[(\partial^r \vec{n})^2 + \mathcal{O}\left(\frac{\partial^{2r+2}}{L^2}\right) \right], \quad (2.4)$$

where

$$d_1 = \frac{1}{8}, \quad d_2 = -\frac{1}{32}, \quad d_3 = \frac{1}{64}, \quad d_4 = -\frac{5}{512}. \quad (2.5)$$

The sum over a in (1.2) becomes an integral over σ as in (2.2), and we find $\sum_{a=1}^L \mathcal{D}_{2r}(\sigma_a) \rightarrow L \left(\int_0^{2\pi} \frac{d\sigma}{2\pi} \mathcal{D}_{2r}(\sigma) + \mathcal{O}\left(\frac{1}{L}\right) \right)$.

2.1 Regularity of the continuum limit

Let us now demand that the same pattern of cancellations (2.4) should persist to all orders in λ -expansion.

Taking the continuum limit in (2.4) one may use the Taylor expansion to show that, up to a total derivative,

$$(\vec{n}_a - \vec{n}_{a+c})^2 = 2 \sum_{m=1}^{\infty} \frac{(-1)^{m-1} (2\pi c)^{2m}}{(2m)! L^{2m}} (\partial^m \vec{n})^2 + \partial(\dots). \quad (2.6)$$

Then

$$\frac{\lambda^r}{(4\pi)^{2r}} \langle \mathcal{D}_{2r}^{(1)} \rangle \rightarrow \lambda^r \sum_{m=1}^{\infty} \frac{d_{rm}}{L^{2m}} (\partial^m \vec{n})^2, \quad (2.7)$$

$$d_{rm} = \frac{(-1)^{m-1} (2\pi)^{2m}}{(4\pi)^{2r} (2m)!} \sum_{c=1}^r c^{2m} a_{r,c}. \quad (2.8)$$

To make sure that the limit (1.13) of $D^{(1)}$ is well defined, $\langle \mathcal{D}_{2r}^{(1)} \rangle$ should scale as $\frac{1}{L^{2r}}$ so that $\lambda^r \langle \mathcal{D}_{2r}^{(1)} \rangle \sim \tilde{\lambda}^r = \frac{\lambda^r}{L^{2r}}$, up to subleading $\mathcal{O}(\frac{1}{L})$ terms. This implies that $d_{r1}, \dots, d_{r,r-1}$

must vanish, i.e. that the coefficients $a_{r,c}$ must satisfy

$$\sum_{c=1}^r c^{2m} a_{r,c} = 0, \quad \text{for } 0 < m < r. \quad (2.9)$$

This gives $(r-1)$ equations for r unknowns. The coefficients $a_{r,c}$ with $c = 2, \dots, r-1$ are then uniquely determined in terms of a single multiplicative constant $a_{r,r}$ from

$$\begin{pmatrix} 1 & 2^2 & \cdots & (r-1)^2 \\ 1 & 2^4 & \cdots & (r-1)^4 \\ & \vdots & \ddots & \vdots \\ 1 & 2^{2(r-1)} & \cdots & (r-1)^{2(r-1)} \end{pmatrix} \begin{pmatrix} a_{r,1} \\ a_{r,2} \\ \vdots \\ a_{r,r-1} \end{pmatrix} = -a_{r,r} \begin{pmatrix} r^2 \\ r^4 \\ \vdots \\ r^{2(r-1)} \end{pmatrix}. \quad (2.10)$$

The matrix in (2.10) is invertible for any r , and one finds⁵

$$a_{r,c} = \frac{(-1)^{r-c}(2r)!}{(r-c)!(r+c)!} a_{r,r}, \quad c = 1, \dots, r-1. \quad (2.11)$$

This generalizes (1.10) to all values of r . Given (2.11), the first non-vanishing coefficient v_r in (2.8) becomes

$$d_{rr} \equiv d_r = \frac{(-1)^{r-1}}{2^{2r}(2r)!} \sum_{c=1}^r c^{2r} a_{r,c} = \frac{(-1)^{r-1}}{2^{2r+1}} a_{r,r}, \quad (2.12)$$

and the r -loop contribution to the expectation value of $D^{(1)}$ takes the form (cf. (2.4))

$$\frac{\lambda^r}{(4\pi)^{2r}} \langle D_{2r}^{(1)} \rangle \rightarrow L \left[d_r \tilde{\lambda}^r \int_0^{2\pi} \frac{d\sigma}{2\pi} (\partial^r \vec{n})^2 + \mathcal{O}\left(\frac{1}{L}\right) \right] \quad (2.13)$$

What remains is to find the values of $a_{r,r}$ generalizing (1.10). This can be done by analyzing the spectrum of BMN operators.

2.2 Constraints from the BMN limit

Let us consider the two-impurity BMN operators of the form

$$O_n^{\text{BMN}} = \frac{1}{\sqrt{J+1}} \sum_{p=0}^J \cos \left[\frac{\pi n(2p+1)}{J+1} \right] \text{tr} (X Z^p X Z^{J-p}). \quad (2.14)$$

⁵The same relation appeared in related context of BMN limit in the first reference in [6]. There the authors computed contributions to the scaling dimensions of the BMN operators by analyzing the relevant Feynman diagrams. Here instead we determine the part of the full dilatation operator which, in particular, computes the dimensions of BMN operators, but which can also be applied systematically to general operators made out of scalars X and Z .

These operators are multiplicatively renormalized for any J , and so are eigenstates of D (see [30] and refs. there). Here the total number of fields, i.e. the length of the spin chain, is $L = J + 2$, with $J_1 \equiv J$ fields Z and $J_2 = 2$ fields X . Anomalous dimensions of the BMN operators can be computed in both string theory and gauge theory [4, 7] in the large J , fixed $\frac{\lambda}{J^2}$ limit, and one finds

$$\Delta_{\text{BMN}} = J + 2\sqrt{1 + \frac{\lambda}{J^2}n^2} + \mathcal{O}\left(\frac{1}{J}\right). \quad (2.15)$$

Note that since $L = J + 2$, this BMN limit is essentially the same as the scaling limit (1.13) discussed above, with $\tilde{\lambda}$ being the same, up to subleading $\frac{1}{J}$ terms, as $\frac{\lambda}{J^2}$ (which is usually denoted as λ'). To reproduce (2.15) (i.e. the coefficients in the expansion of the square root in powers of $\tilde{\lambda}$) by acting with the dilatation operator (1.1) on O_n^{BMN} , we should require that

$$D_{2r}O_n^{\text{BMN}} = 4\frac{(-1)^{r-1}\Gamma(2r-1)}{\Gamma(r)\Gamma(r+1)}\left(\frac{2\pi n}{J}\right)^{2r}O_n^{\text{BMN}} + \mathcal{O}\left(\frac{1}{J^{2r+1}}\right). \quad (2.16)$$

To evaluate $D_{2r}O_n^{\text{BMN}}$ explicitly let us note since we are interested in the large J limit, we can ignore what happens near the ends of the spin chain. Then for generic values of p in the sum (2.14) we find that the permutation operators $P_{a,b} = \mathbf{1} - Q_{a,b}$ act as follows (c can be positive or negative)

$$P_{1,1+c}O_p = P_{p+2,p+2-c}O_p = O_{p-c}, \quad O_p \equiv \text{tr}(XZ^pXZ^{J-p}), \quad (2.17)$$

while for all other labels a, b we get $P_{a,b}O_p = O_p$. Then

$$\sum_{a=1}^{J+2} Q_{a,a+c}O_p = 2(O_p - O_{p-c}) + 2(O_p - O_{p+c}), \quad (2.18)$$

where there are four nonzero combinations, from $Q_{1,1\pm|c|}$ and from $Q_{p+2,p+2\pm|c|}$. Other terms $\mathcal{D}_{2r}^{(n)}$ in D_{2r} (1.8) containing two and more products of Q 's (which appear starting with 3-loop term (1.6)), i.e. $Q_{a,b}\dots Q_{c,d}$ with all labels a, b, \dots, c, d distinct, annihilate O_p unless we are near the ends of the chain, i.e. they do not contribute to (2.16) in the large J limit. Ignoring these higher order terms is equivalent to the usual dilute gas approximation which applies when the number of impurities is small and the spin chain is very long. Thus it is the $D^{(1)}$ part of the full dilatation operator (1.11) that is responsible for the anomalous dimensions of the operators O_n^{BMN} in the BMN limit,

$$(D - D_0)O_n^{\text{BMN}} = D^{(1)}O_n^{\text{BMN}} + \dots, \quad (2.19)$$

where dots stand for subleading $\frac{1}{J}$ terms. Using (2.18) we find that to the leading order in $\frac{1}{J}$,

$$\begin{aligned} D_{2r}^{(1)} O_n^{\text{BMN}} &= \sum_{c=1}^r a_{r,c} \frac{4}{\sqrt{J+1}} \sum_{p=0}^J \cos \left[\frac{\pi n(2p+1)}{J+1} \right] (2O_p - O_{p-c} - O_{p+c}) \\ &= 8 \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m)!} \left(\frac{2\pi n}{J} \right)^{2m} \sum_{c=1}^r c^{2m} a_{r,c} O_n^{\text{BMN}} . \end{aligned} \quad (2.20)$$

To arrive at this expression we expanded the cosines at large J . The sum $\sum_{c=1}^r c^{2m} a_{r,c}$ in (2.20) is precisely the same as in (2.7),(2.8). Using the results (2.9) and (2.12) for $a_{r,c}$,

$$\begin{aligned} D_{2r} O_n^{\text{BMN}} &= 4(-1)^{r-1} a_{r,r} \left(\frac{2\pi n}{J} \right)^{2r} O_n^{\text{BMN}} + \mathcal{O}\left(\frac{1}{J^{2r+1}}\right) \\ &= (-1)^{r-1} a_{r,r} (D_2)^r O_n^{\text{BMN}} + \mathcal{O}\left(\frac{1}{J^{2r+1}}\right) , \end{aligned} \quad (2.21)$$

so to match the BMN expression (2.16) we must have

$$a_{r,r} = \frac{\Gamma(2r-1)}{\Gamma(r)\Gamma(r+1)} = \frac{2^{2r-3}\Gamma(r-\frac{1}{2})}{\sqrt{\pi}\Gamma(r+1)} . \quad (2.22)$$

For $r = 1, 2, 3, 4$ this gives $a_{r,r} = 1, 1, 2, 5$ as in (1.10) (note that the limit of $r \rightarrow 1$ of the second expression in (2.22) is well defined). Combining (2.11) with (2.22) we finally conclude that

$$a_{r,c} = \frac{(-1)^{r-c} \Gamma(2r+1) \Gamma(2r-1)}{\Gamma(r-c+1) \Gamma(r+c+1) \Gamma(r) \Gamma(r+1)} . \quad (2.23)$$

Note that, as required (cf. (1.9)), $a_{r,c} = 0$ for $r < c$ (for integer r and c).

3 Summing up: structure of $D^{(1)}$ to all orders in λ

Let us now try to draw some conclusions about the exact structure of the “spin-spin” part of the dilatation operator $D^{(1)}$ in (1.11) to all orders in λ . First, let us observe that the value of $a_{r,r}$ found in (2.22) implies that summing (2.13) over r with d_r in (2.12) gives a very simple formula for the quadratic in \vec{n} (“small fluctuation” or “BMN”) part of the coherent state effective Hamiltonian in (2.2):

$$\langle D^{(1)} \rangle = \sum_{r=1}^{\infty} \frac{\lambda^r}{(4\pi)^{2r}} \langle D_{2r}^{(1)} \rangle \rightarrow L \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[\frac{1}{4} \vec{n} \left(\sqrt{1 - \tilde{\lambda} \partial^2} - 1 \right) \vec{n} + \mathcal{O}\left(\frac{1}{L}\right) \right] . \quad (3.1)$$

Remarkably, this is the same expression that follows from the classical $AdS_5 \times S^5$ string sigma model action expanded in the limit (1.13) (eq. (2.90) in [21]). There is a closely

related square root formula (see eq. (3.25) in [21]) that expresses $D^{(1)}$ as a function of D_2 in the dilute gas approximation:

$$D^{(1)} = \left(\sqrt{1 + 2 \frac{\lambda}{(4\pi)^2} D_2} - 1 \right)^{(1)}. \quad (3.2)$$

The superscript “(1)” in the right hand side means that one should drop all terms with higher than first power of independent $Q_{a,b}$ ’s (written in the factorized form) in the products of D_2 . This relation should be understood in the sense of equality of the \vec{n}^2 terms in the coherent-state expectation values of the two sides.⁶

Next, let us substitute the values (2.23) for the coefficients $a_{r,c}$ we have found above into $D^{(1)}$ in (1.11) and try to formally perform the summation over r first, independently for each $Q_{a,a+c}$ term. We get

$$D^{(1)} = 2 \sum_{a=1}^L \sum_{c=1}^{\infty} f_c(\lambda) Q_{a,a+c}, \quad f_c(\lambda) = \sum_{r=c}^{\infty} \frac{\lambda^r}{(4\pi)^{2r}} a_{r,c}. \quad (3.3)$$

Remarkably, the series representation for the coefficients $f_c(\lambda)$ can then be summed up in terms of the standard hypergeometric functions (see Appendix):

$$\begin{aligned} f_c(\lambda) &= \sum_{r=c}^{\infty} \frac{\lambda^r}{(4\pi)^{2r}} \frac{(-1)^{r-c} \Gamma(2r-1) \Gamma(2r+1)}{\Gamma(r+1) \Gamma(r) \Gamma(r-c+1) \Gamma(r+c+1)} \\ &= \left(\frac{\lambda}{4\pi^2} \right)^c \frac{\Gamma(c-\frac{1}{2})}{4\sqrt{\pi} \Gamma(c+1)} {}_2F_1\left(c-\frac{1}{2}, c+\frac{1}{2}; 2c+1; -\frac{\lambda}{\pi^2}\right). \end{aligned} \quad (3.4)$$

The coefficient in front of ${}_2F_1$ is equal to $\frac{2\lambda^c}{(4\pi)^{2c}} a_{c,c}$ (cf. (2.22)). The f_c go to 0 rapidly at large c , so we effectively have a spin chain with a short range interactions.

In general, the hypergeometric functions ${}_2F_1(a_1, a_2; b_1; z)$ have a cut in the z plane running from 1 to ∞ . Note also that $y(z) = {}_2F_1(c-\frac{1}{2}, c+\frac{1}{2}; 2c+1; z)$ solves the following differential equation $z(1-z)y'' + (2c+1)(1-z)y' - (c^2 - \frac{1}{4})y = 0$.

The resulting coefficients $f_c(\lambda)$ are smooth positive functions of λ having regular expansion at both small λ (see (3.4)) and large λ

$$f_c(\lambda)_{\lambda \rightarrow \infty} = \frac{\sqrt{\lambda}}{\pi^2} \left[\frac{1}{4c^2-1} - \frac{\pi^2}{4\lambda} \left(\ln \frac{\lambda}{\pi^2} + 1 - 2H_{c-\frac{1}{2}} \right) + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \right] \quad (3.5)$$

⁶If we wanted to apply $D^{(1)}$ to a single trace operator, the compact expression in the right hand side of (3.2) would be of limited advantage. Multiplying D_2 ’s and taking the single $Q_{a,b}$ part do not commute, and we would have to explicitly compute all powers of D_2 and then apply the “(1)-operation” to them (like it was done in Appendix C of [21] where connected expectation values of products of some operators were computed).

where the harmonic numbers H_p are defined in Appendix. The square root $f_c \rightarrow \sqrt{\lambda}$ asymptotics of (3.5) is related to the cut structure of ${}_2F_1$. The $\ln \lambda$ subleading terms are likely to be an artifact of our resummation procedure (they will be absent in the explicit $L = 4$ example discussed below).

One may be tempted to interpret the behavior of the coefficients f_c as an indication of how anomalous dimensions of particular operators should scale with λ (one should remember of course that $D^{(1)}$ is only a part of the full dilatation operator in (1.11)). Their $\sqrt{\lambda}$ asymptotics may then seem to be in contradiction with the usual expectation that dimensions of generic operators corresponding to string modes should have slower growth with λ – they should scale as square root of the effective string tension, i.e. as $\sqrt[4]{\lambda}$ [5].

A possible resolution of this paradox is that the above resummation procedure leading to (3.3),(3.4) is useful only in the infinite chain $L \rightarrow \infty$ limit, i.e. it corresponds to the case when D acts on “long” operators. The latter should be dual to semiclassical string modes for which dimensions are expected to grow as string tension $\sim \sqrt{\lambda}$ [5]. Indeed, we have treated all $Q_{a,a+c}$ terms as independent but for finite L the terms with c and $c + mL$ are the same because of the periodicity of the chain (implied by cyclicity of the trace in the operators). Also, under the sum over a one has $Q_{a,a+c} = Q_{a,a+L-c}$, i.e.

$$\Pi_c = \Pi_{c+mL} = \Pi_{mL-c} , \quad \Pi_c \equiv \sum_{a=1}^L Q_{a,a+c} , \quad (3.6)$$

where m is any positive integer number. Therefore, for finite L the sum over c should, in fact, be restricted to run from $c = 1$ to $c = L - 1$,

$$D^{(1)} = 2 \sum_{a=1}^L \sum_{c=1}^{\infty} f_c(\lambda) Q_{a,a+c} = \sum_{a=1}^L \sum_{c=1}^{L-1} h_c(L, \lambda) Q_{a,a+c} , \quad (3.7)$$

or, equivalently,

$$D^{(1)} = 2 \sum_{c=1}^{[L/2]} h_c(L, \lambda) \Pi_c . \quad (3.8)$$

The new coefficients h_c depend on both the 't Hooft coupling λ and the length of the chain L ,

$$\begin{aligned} h_c(L, \lambda) &\equiv \sum_{m=0}^{\infty} [f_{c+mL}(\lambda) + f_{L-c+mL}(\lambda)] , & c = 1, \dots, L-1; \ c \neq L/2 \\ h_{L/2}(L, \lambda) &\equiv \sum_{m=0}^{\infty} f_{L/2+mL}(\lambda) , & \text{if } L \text{ is even} . \end{aligned} \quad (3.9)$$

They satisfy the periodicity condition $h_c(L, \lambda) = h_{L-c}(L, \lambda)$, reflecting the fact that for pairwise interactions it matters only which sites participate in the interaction. Explicitly, the coefficients appearing in (3.9) can be written using (3.4) as

$$h_c(L, \lambda) = \sum_{m=0}^{\infty} \left[\left(\frac{\lambda}{4\pi^2} \right)^{c+mL} \frac{\Gamma(c+mL-\frac{1}{2})}{4\sqrt{\pi} \Gamma(c+mL+1)} \right. \\ \left. \times {}_2F_1\left(c+mL-\frac{1}{2}, c+mL+\frac{1}{2}; 2c+2mL+1; -\frac{\lambda}{\pi^2}\right) \right] \\ + (c \rightarrow L-c) . \quad (3.10)$$

When $L \rightarrow \infty$ and $0 < c \ll L$, the only contribution to the sum (3.10) comes from $m = 0$ in the first term, and (3.10) reduces to (3.4). For finite L , we may expand the hypergeometric functions in (3.10) at large λ as in (3.5) and then do the sum over m . Ignoring the issue of convergence of the resulting strong-coupling expansion, that leads to the following simple result for the leading-order term

$$h_c(L, \lambda)_{\lambda \rightarrow \infty} = \frac{\sqrt{\lambda}}{2\pi L} \left[\frac{\sin \frac{\pi}{L}}{\cos \frac{\pi}{L} - \cos \frac{2\pi c}{L}} + \mathcal{O}(\lambda^{-1}) \right]. \quad (3.11)$$

In the $c \ll L$ limit (3.11) reduces back to the large λ asymptotics (3.5) of the $m = 0$ term of the sum in (3.10).

It is possible that for finite L the contributions of higher order Q^n interaction terms in D (1.11) may transform this asymptotics into the expected $\sqrt[4]{\lambda}$ behavior. At the same time, one may wonder if our basic assumption about the structure of $\mathcal{D}_{2r}^{(1)}$ in (1.9) actually applies for finite values of L and all values of r . After all, to fix the coefficients $a_{r,c}$ we used the condition of regularity of the scaling limit which assumes that $L \rightarrow \infty$.⁷

Ignoring this cautionary note let us go ahead and apply the above relations to the first non-trivial small L case – $L = 4$ (the operators with lengths $L = 2, 3$ are BPS: the antisymmetric combinations vanish because of trace cyclicity). The non-BPS operator with $L = 4$ is the level four descendant K of the Konishi scalar operator

$$K = \text{tr} [X, Z]^2 = 2 \text{tr} (XZXZ - XXZZ) . \quad (3.12)$$

The action of $D^{(1)}$ (3.7) on K is determined by noting that (see (3.6))

$$\Pi_1 K = \Pi_3 K = 6K, \quad \Pi_2 K = 0, \quad D^{(1)} K = \gamma^{(1)} K . \quad (3.13)$$

⁷Possible subtleties in applying the general expression for the dilatation operator to operators with small length L were mentioned in [23] (footnote 18), [24] (footnote 4) and [25] (section 4.3).

Using that the periodicity implies that $\Pi_1 = \Pi_3 = \Pi_5 = \Pi_7 = \dots$ one finds then directly from (1.11)

$$\gamma^{(1)} = \sum_{k=1}^{\infty} \sum_{p=1}^k \left[\left(\frac{\lambda}{16\pi^2} \right)^{2k} a_{2k,2p-1} + \left(\frac{\lambda}{16\pi^2} \right)^{2k-1} a_{2k-1,2p-1} \right]. \quad (3.14)$$

Using (2.23) the sums over p can be found explicitly

$$\sum_{p=1}^k a_{2k,2p-1} = -\frac{2^{4k-2}\Gamma(4k-1)}{\Gamma(2k)\Gamma(2k+1)}, \quad \sum_{p=1}^k a_{2k-1,2p-1} = \frac{2^{4k-4}\Gamma(4k-3)}{\Gamma(2k)\Gamma(2k-1)}, \quad (3.15)$$

and finally we obtain the following surprisingly simple result (cf. (A.2))

$$\gamma^{(1)} = \frac{3}{2} \left(\sqrt{1 + \frac{\lambda}{\pi^2}} - 1 \right). \quad (3.16)$$

Then for small λ we reproduce the previously known results [31, 23]

$$\gamma^{(1)} = 3\frac{\lambda}{4\pi^2} - 3\left(\frac{\lambda}{4\pi^2}\right)^2 + 6\left(\frac{\lambda}{4\pi^2}\right)^3 - 15\left(\frac{\lambda}{4\pi^2}\right)^4 + \mathcal{O}(\lambda^5), \quad (3.17)$$

while in the large λ limit one finds

$$\gamma^{(1)} = \frac{3}{2\pi}\sqrt{\lambda} - \frac{3}{2} + \frac{3\pi}{2\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{\lambda^{3/2}}\right), \quad (3.18)$$

where the leading term agrees with (3.11). Note also that in the λ^3 and λ^4 terms in (3.17) we included only the contributions of $D_6^{(1)}$ and $D_8^{(1)}$, i.e. the linear in Q terms in (1.6) and (1.7). The contributions of the QQ terms in (1.6) and (1.7) change the 3-loop and 4-loop coefficients in (3.17) from 6 to $\frac{21}{4}$ [23] and from -15 to $-\frac{705}{64}$ [24] (but see footnote 7). Again, one may hope that a systematic account of contributions of all higher $\mathcal{D}^{(n)}$ terms in (1.8) will change the strong-coupling asymptotics of the dimension of the Konishi operator from $\sqrt{\lambda}$ to $\sqrt[4]{\lambda}$.

4 Concluding remarks

Inspired by recent work in [23, 24] and especially [19], in this paper we suggested to organize the dilatation operator as an expansion (1.11), (1.8) in powers of independent projection operators $Q_{a,b}$ [19, 21] at L sites of spin chain

$$D = D_0 + \sum_{n=1}^{\infty} D^{(n)} \quad \text{with } D_0 = L, \quad D^{(1)} = \sum_{a=1}^L \sum_{c=1}^{L-1} h_c(L, \lambda) Q_{a,a+c}, \dots \quad (4.1)$$

where $D^{(n)}$ are given by sums of products of n Q 's at independent sites of the spin chain. We determined the coefficients in $D^{(1)}$ by demanding that its BMN-type scaling limit [18, 21] be regular, and found that it admits a very simple representation (3.3), applicable at least in the large L limit. This representation includes all orders in λ and suggests that the corresponding anomalous dimensions should grow as $\sqrt{\lambda}$ for large λ .

A natural extension of this work would be to try to find the next term in the expansion (4.1), namely

$$D^{(2)} = \sum_{a=1}^L \sum_{c_1, c_2, c_3=1}^{L-1} ' h_{c_1; c_2, c_3}(L, \lambda) Q_{a, a+c_1} Q_{a+c_2, a+c_2+c_3} . \quad (4.2)$$

The prime on the sum here means that certain terms should be omitted: since $Q_{a,b}Q_{b,c} + Q_{b,c}Q_{a,b} = Q_{a,b} + Q_{b,c} - Q_{a,c}$ the terms with $c_2 = c_1$ and $c_2 = c_3 - c_1$ have already been included in $D^{(1)}$. The contributions of higher $D^{(n)}$ terms should be crucial at finite L , resolving, in particular, the above-mentioned contradiction between the $\sqrt{\lambda}$ asymptotics of the coefficients in $D^{(1)}$ and the expected $\sqrt[4]{\lambda}$ scaling of dimensions of operators corresponding to string modes.

The AdS/CFT duality suggests that D should correspond to an integrable spin chain. The simplest possibility could be that, by analogy with the Inozemtsev chain [19], the operator $D^{(1)}$ (with interaction coefficients given by (3.10)) represents a Hamiltonian of an integrable spin 1/2 chain, while all higher order terms $D^{(n)}$ are effectively determined by $D^{(1)}$ through integrability. This, however, is unlikely in view of the low loop order results of [23, 24, 25] and the very recent paper [32] suggesting that the *full* dilatation operator satisfying the requirements of integrability, BMN scaling and consistency with gauge theory should be essentially unique.

5 Acknowledgments

We are grateful to G. Arutyunov, N. Beisert, S. Frolov, M. Kruczenski, A. Parnachev, A. Onishchenko, M. Staudacher and K. Zarembo for useful discussions, e-mail correspondence on related issues and comments on an earlier version of this paper. The work of A.R. was supported in part by NSF grants PHY99-73935 and PHY04-01667. The work of A.T. was supported by DOE grant DE-FG02-91ER40690, the INTAS contract 03-51-

6346 and Wolfson award. We also thank the Michigan Center for Theoretical Physics for the hospitality during the beginning of this work.

Appendix A Some useful relations and definitions

Here we summarize some useful formulae used in the paper. The usual binomial expansion is given by

$$(1+x)^n = \sum_{k=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} x^k \quad (\text{A.1})$$

(for integer n the series terminates at $k = n$). For $n = \frac{1}{2}$ (A.1) can be written as

$$(1+x)^{1/2} = 1 + 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \Gamma(2k-1)}{\Gamma(k)\Gamma(k+1)} \left(\frac{x}{4}\right)^k. \quad (\text{A.2})$$

To transform the arguments of Γ -functions one uses

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z + \frac{1}{2}), \quad \Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin \pi z}. \quad (\text{A.3})$$

The hypergeometric functions are given, within the radius of convergence, by the series

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!}, \quad (a)_k \equiv \frac{\Gamma(a+k)}{\Gamma(a)}. \quad (\text{A.4})$$

They reduce to simpler functions in particular cases; for example ${}_2F_1(-n, a; a, -z) = (1+z)^n$ as one can see by comparing (A.1) and (A.4). If $p = q+1$, ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z)$ have a branch cut in the z -plane running from $z = 1$ to ∞ . ${}_pF_q$ satisfy second order differential equations, and by a change of variables one can relate the values of ${}_pF_q$ at z and at $1/z$, although for different arguments a_i and b_j . One can show that for large z

$${}_2F_1(b-1, b; 2b; z) = \frac{(-z)^{1-b} \Gamma(2b)}{\Gamma(b) \Gamma(b+1)} \left[1 + \frac{b(b-1)}{(-z)} \left(1 + \ln(-z) - 2H_{b-\frac{1}{2}} \right) + \mathcal{O}(z^{-2}) \right], \quad (\text{A.5})$$

where the Harmonic numbers are given by

$$H_p \equiv H_p^{(1)}, \quad H_p^{(s)} = \zeta(s, 1) - \zeta(s, p+1), \quad \zeta(s, p) \equiv \sum_{k=0}^{\infty} (k+p)^{-s}, \quad (\text{A.6})$$

and for integer $p > 1$ one has $H_p^{(s)} = \sum_{k=1}^p k^{-s}$. One finds that $H_{b-\frac{1}{2}} = \gamma_E + \psi(b)$, where $\psi(b) = \frac{\Gamma'(b)}{\Gamma(b)}$, and $\gamma_E = -\psi(1) \approx 0.577216$ is the Euler's constant.

References

- [1] F. Gliozzi, J. Scherk and D. I. Olive, “Supersymmetry, Supergravity Theories And The Dual Spinor Model,” Nucl. Phys. B **122**, 253 (1977). L. Brink, J. H. Schwarz and J. Scherk, “Supersymmetric Yang-Mills Theories,” Nucl. Phys. B **121**, 77 (1977).
- [2] M. F. Sohnius and P. C. West, “Conformal Invariance In N=4 Supersymmetric Yang-Mills Theory,” Phys. Lett. B **100**, 245 (1981). L. Brink, O. Lindgren and B. E. W. Nilsson, “The Ultraviolet Finiteness Of The N=4 Yang-Mills Theory,” Phys. Lett. B **123**, 323 (1983). P. S. Howe, K. S. Stelle and P. K. Townsend, “Miraculous Ultraviolet Cancellations In Supersymmetry Made Manifest,” Nucl. Phys. B **236**, 125 (1984).
- [3] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. **2**, 231 (1998) [hep-th/9711200]. S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B **428**, 105 (1998) [hep-th/9802109]. E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. **2**, 253 (1998) [hep-th/9802150].
- [4] D. Berenstein, J. M. Maldacena and H. Nastase, “Strings in flat space and pp waves from N =4 super Yang Mills,” JHEP **0204**, 013 (2002) [hep-th/0202021].
- [5] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “A semi-classical limit of the gauge/string correspondence,” Nucl. Phys. B **636**, 99 (2002) [hep-th/0204051].
- [6] D. J. Gross, A. Mikhailov and R. Roiban, “Operators with large R charge in N = 4 Yang-Mills theory,” Annals Phys. **301**, 31 (2002) [hep-th/0205066]. “A calculation of the plane wave string Hamiltonian from N = 4 super-Yang-Mills theory,” JHEP **0305**, 025 (2003) [hep-th/0208231].
- [7] A. Santambrogio and D. Zanon, “Exact anomalous dimensions of N = 4 Yang-Mills operators with large R charge,” Phys. Lett. B **545**, 425 (2002) [hep-th/0206079].
- [8] S. Frolov and A. A. Tseytlin, “Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$,” JHEP **0206**, 007 (2002) [hep-th/0204226].
- [9] J. G. Russo, “Anomalous dimensions in gauge theories from rotating strings in $AdS(5) \times S(5)$,” JHEP **0206**, 038 (2002) [hep-th/0205244].
- [10] J. A. Minahan, “Circular semiclassical string solutions on $AdS_5 \times S^5$,” Nucl. Phys. B **648**, 203 (2003) [hep-th/0209047].

- [11] J. A. Minahan and K. Zarembo, “The Bethe-ansatz for $N = 4$ super Yang-Mills,” JHEP **0303**, 013 (2003) [hep-th/0212208].
- [12] S. Frolov and A. A. Tseytlin, “Multi-spin string solutions in $AdS_5 \times S^5$,” Nucl. Phys. B **668**, 77 (2003) [hep-th/0304255].
- [13] N. Beisert, J. A. Minahan, M. Staudacher and K. Zarembo, “Stringing spins and spinning strings,” JHEP **0309**, 010 (2003) [hep-th/0306139].
- [14] S. Frolov and A. A. Tseytlin, “Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors,” Phys. Lett. B **570**, 96 (2003) [hep-th/0306143].
- [15] N. Beisert, S. Frolov, M. Staudacher and A. A. Tseytlin, “Precision spectroscopy of AdS/CFT,” JHEP **0310**, 037 (2003) [hep-th/0308117].
- [16] G. Arutyunov and M. Staudacher, “Matching higher conserved charges for strings and spins,” JHEP **0403**, 004 (2004) [hep-th/0310182]. “Two-loop commuting charges and the string / gauge duality,” hep-th/0403077.
- [17] J. Engquist, J. A. Minahan and K. Zarembo, “Yang-Mills duals for semiclassical strings on $AdS_5 \times S^5$,” JHEP **0311**, 063 (2003) [hep-th/0310188].
- [18] M. Kruczenski, “Spin chains and string theory,” hep-th/0311203.
- [19] D. Serban and M. Staudacher, “Planar $N = 4$ gauge theory and the Inozemtsev long range spin chain,” hep-th/0401057.
- [20] V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, “Classical/quantum integrability in AdS/CFT,” hep-th/0402207.
- [21] M. Kruczenski, A. V. Ryzhov and A. A. Tseytlin, “Large spin limit of $AdS_5 \times S^5$ string theory and low energy expansion of ferromagnetic spin chains,” hep-th/0403120.
- [22] A. V. Kotikov, L. N. Lipatov and V. N. Velizhanin, “Anomalous dimensions of Wilson operators in $N = 4$ SYM theory,” Phys. Lett. B **557**, 114 (2003) [hep-ph/0301021].
A. V. Kotikov, L. N. Lipatov, A. I. Onishchenko and V. N. Velizhanin, “Three-loop universal anomalous dimension of the Wilson operators in $N = 4$ SUSY Yang-Mills model,” hep-th/0404092.
- [23] N. Beisert, C. Kristjansen and M. Staudacher, “The dilatation operator of $N = 4$ super Yang-Mills theory,” Nucl. Phys. B **664**, 131 (2003) [hep-th/0303060].

- [24] N. Beisert, “Higher loops, integrability and the near BMN limit,” JHEP **0309**, 062 (2003) [hep-th/0308074].
- [25] N. Beisert, “The $su(2|3)$ dynamic spin chain,” hep-th/0310252.
- [26] T. Klose and J. Plefka, “On the integrability of large N plane-wave matrix theory,” Nucl. Phys. B **679**, 127 (2004) [hep-th/0310232].
- [27] F.D.M. Haldane, “Exact Jastrow-Gutzwiller Resonating Valence Bond Ground State Of The Spin $1/2$ Antiferromagnetic Heisenberg Chain With $1/R^2$ Exchange”, Phys. Rev. Lett. **60**, 635 (1988); S. Shastry, “Exact Solution Of An $S = 1/2$ Heisenberg Antiferromagnetic Chain With Long Ranged Interactions”, Phys. Rev. Lett. **60**, 639 (1988).
- [28] V. I. Inozemtsev, “On The Connection Between The One-Dimensional $S = 1/2$ Heisenberg Chain And Haldane Shastry Model,” J. Stat. Phys. **50**, 1143 (1990); “Integrable Heisenberg-van Vleck chains with variable range exchange,” Phys. Part. Nucl. **34**, 166 (2003) [Fiz. Elem. Chast. Atom. Yadra **34**, 332 (2003)] [hep-th/0201001].
- [29] E. H. Fradkin, “Field Theories Of Condensed Matter Systems,” Redwood City, USA: Addison-Wesley (1991) 350 p. (Frontiers in physics, 82).
- [30] N. Beisert, “BMN operators and superconformal symmetry,” Nucl. Phys. B **659**, 79 (2003) [hep-th/0211032].
- [31] D. Anselmi, “The $N = 4$ quantum conformal algebra,” Nucl. Phys. B **541**, 369 (1999) [hep-th/9809192].
- [32] N. Beisert, V. Dippel and M. Staudacher, “A Novel Long Range Spin Chain and Planar $N=4$ Super Yang-Mills”, hep-th/0405001.